Part 3.2 Differentiation

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Derivative Results

We now come to the three main results of this section on Differentiation. The first, Rolle's Theorem is a special case of the second, the Mean Value Theorem. The third result is Taylor's Theorem which generalizes the Mean Value Theorem to include higher derivatives.

Definition 3.2.1 *Let $f : A \to \mathbb{R}$.*

- *We say that f has a **minimum at** $k \in A$ if $f(k) \leq f(x)$ for all $x \in A$.*
- *We say that f has a **maximum at** $\ell \in A$ if $f(x) \leq f(\ell)$ for all $x \in A$.*
- *We say that f has an **extremum** at a if a is either a minimum or maximum.*

Definition 3.2.2 *Let $f : A \to \mathbb{R}$.*

- *We say that f has a **local minimum at** $k \in A$ if $f(k) \leq f(x)$ for all x in some open neighbourhood of k, i.e.*

  $$\exists\, \delta > 0 : (k - \delta, k + \delta) \subseteq A \quad \textit{and} \quad \forall x : x \in (k - \delta, k + \delta) \implies f(k) \leq f(x)\,.$$

- *We say that f has a **local maximum at** $\ell \in A$ if $f(x) \leq f(\ell)$ for all x in some open neighbourhood of k, i.e.*

  $$\exists\, \delta > 0 : (\ell - \delta, \ell + \delta) \subseteq A \quad \textit{and} \quad \forall x : x \in (\ell - \delta, \ell + \delta) \implies f(x) \leq f(\ell)\,.$$

- *We say that f has a **local extremum** at a if a is either a local minimum or local maximum.*

Theorem 3.2.3 *If $f : A \to \mathbb{R}$ has a local extremum at $a \in \mathbb{R}$ and f is differentiable at a then $f'(a) = 0$.*

Proof Assume f has a local maximum at a. By the definition

$$\exists\, \delta > 0 : \forall x \in A, a - \delta < x < a + \delta \implies f(x) \leq f(a)\,.$$

By assumption f is differentiable at a so the limit of $f(x) - f(a) / (x - a)$ as $x \to a$ exists which implies the two one-sided limits exit. Thus we have **two cases**, the left hand limit and the right hand limit.

First case: x satisfying $a-\delta < x < a$. Then $f(x)-f(a) \leq 0$ and $x-a<0$. The quotient of two negative numbers is positive so

$$\frac{f(x)-f(a)}{x-a} \geq 0.$$

Thus

$$\lim_{x\to a-} \frac{f(x)-f(a)}{x-a} \geq 0.$$

Second case: x satisfying $a < x < a+\delta$. Then $f(x)-f(a) \leq 0$ and $x-a>0$. The quotient of a negative number by a positive is negative so

$$\frac{f(x)-f(a)}{x-a} \leq 0.$$

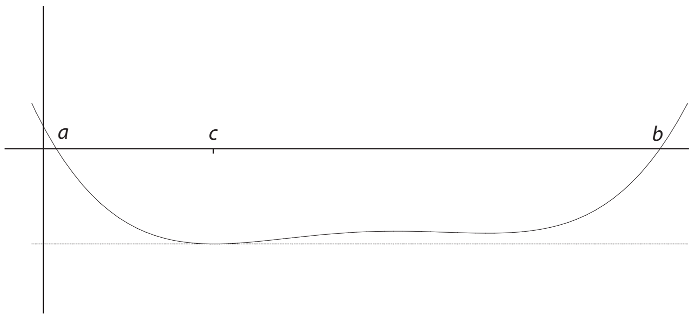
Hence

$$\lim_{x\to a+} \frac{f(x)-f(a)}{x-a} \leq 0.$$

Since f is differentiable at a the one-sided limits not only exist *but are equal*. The **only** number that is both ≥ 0 and ≤ 0 is 0, hence $f'(a)=0$.

If f has a local minimum at a apply the above to $-f$. ■

Theorem 3.2.4 ***Rolle's Theorem*** *(1690) If f is differentiable on the open interval (a,b), continuous on the closed interval $[a,b]$ and $f(a)=f(b)$ then there exists a $c : a < c < b$ such that $f'(c)=0$.*



Proof Since f is continuous on the closed and bounded interval $[a,b]$ it is bounded and its bounds are attained. So there exist $k,\ell \in [a,b]$ such that $f(k) \leq f(x) \leq f(\ell)$ for all $x \in [a,b]$.

If $f(k)=f(\ell)$ then $f(x)$ equals this common value for all x thus f is constant. Hence $f'(x)=0$ for all $x \in [a,b]$. Choose $c=(a+b)/2$.

Otherwise $f(k) < f(\ell)$. Thus at least one of $f(k)$ and $f(\ell)$ differs from the common value of $f(a) = f(b)$. Assume the minimal value $f(k)$ differs from $f(a)$ and $f(b)$. Then $f(k) \neq f(a)$ means that $k \neq a$ while $f(k) \neq f(b)$ implies $k \neq b$. Thus $k \in (a, b)$, i.e. k is in the open interval. This means that f is differentiable at k. Then $f(k) \leq f(x)$ for all $x \in (a, b)$ means that k is a minimum of f and thus, by Theorem 3.2.3, satisfies $f'(k) = 0$.

I leave it to the interested student to show that if the maximal value $f(\ell)$ differs from $f(a)$ and $f(b)$ then $f'(\ell) = 0$. ■

We previously used the Intermediate Value Theorem to show the *existence* of roots of some equations. We can now use Rolle's Theorem to say something about the *uniqueness* of these roots.

Example 3.2.5 *Show that*

$$\cos x + 2\cos\left(\frac{x}{2}\right) = \sin x + 2\sin\left(\frac{x}{2}\right)$$

has exactly one solution in $[0, \pi/2]$.

Solution Let

$$f(x) = \cos x + 2\cos\left(\frac{x}{2}\right) - \sin x - 2\sin\left(\frac{x}{2}\right).$$

This is everywhere continuous and thus continuous on $[0, \pi/2]$. Also $f(0) = 3$ and $f(\pi/2) = -1$. So 0 is an intermediate value and thus, by the Intermediate Value Theorem there exists $c \in (0, \pi/2)$ such that $f(c) = 0$.

Assume now this is not unique. Let $c_1, c_2 \in (0, \pi/2)$, $c_1 < c_2$ be two such solutions. Consider f on $[c_1, c_2]$. The function f is everywhere differentiable and thus differentiable on (c_1, c_2). Also $f(c_1) = f(c_2) = 0$ and so, by Rolle's Theorem there exists $c \in (c_1, c_2) : f'(c) = 0$. Yet

$$f'(x) = -\sin x - \sin\left(\frac{x}{2}\right) - \cos x - \cos\left(\frac{x}{2}\right) < 0$$

for all $x \in (0, \pi/2)$. Contradiction, last assumption is false, i.e. solution is unique. ■

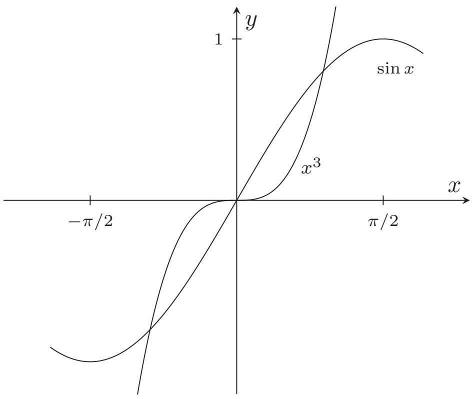
Example 3.2.6 *Show that*

$$\sin x = x^3$$

has ***exactly*** *three solutions in* $\mathbb{R}$.

Solution left for student (and Tutorial). ■

The illustration of Example 3.2.6 is:

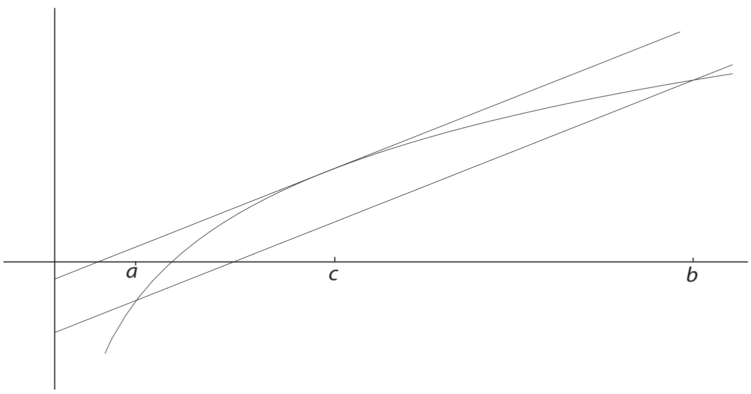


The next result generalises Rolle's Theorem in that it removes the requirement that $f(a) = f(b)$.

Theorem 3.2.7 ***Mean Value Theorem*** *(Lagrange 1797) If the function f is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$ then there exists $c : a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Graphically we can draw a tangent at some point c with slope the same as that of the chord joining points $(a, f(a))$ to $(b, f(b))$:



Proof Define $h(x) = f(x) - \kappa x$ with κ chosen to make $h(a) = h(b)$. That is,

$$f(a) - \kappa a = f(b) - \kappa b,$$

which rearranges as

$$\kappa = \frac{f(b) - f(a)}{b - a}.$$

The new function h is still differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$. So we can apply Rolle's Theorem to h on $[a, b]$ to find $c : a < c < b$ such that $h'(c) = 0$. That is

$$f'(c) = \kappa = \frac{f(b) - f(a)}{b - a}.$$

■

Note If $f(b) = f(a)$ we would have found a c satisfying $f'(c) = 0$, which is Rolle's Theorem. (Though, of course, the proof of the Mean Value Theorem **makes use of** Rolle's Theorem, so we do not get a new proof of Rolle's Theorem.)

Applications By bounding $|f'(c)|$ we can give inequalities between $f(a)$ and $f(b)$.

Example 3.2.8 *(Application of the Mean Value Theorem) Without using a calculator, show that*

$$\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}.$$

Solution Consider $f(x) = \sqrt{x}$ on the interval $[64, 66]$. The function f is continuous on $[64, 66]$ and differentiable on $(64, 66)$ so we can apply the Mean Value Theorem to find $c : 64 < c < 66$ such that

$$\sqrt{66} - \sqrt{64} = f(66) - f(64) = f'(c)(66 - 64) = \frac{1}{2\sqrt{c}}2 = \frac{1}{\sqrt{c}}.$$

Thus, since $64 < c < 66$,

$$\frac{1}{\sqrt{66}} < \sqrt{66} - \sqrt{64} < \frac{1}{\sqrt{64}}. \tag{1}$$

To simplify the result use $c < 66 < 81$ so the lower bound in (1) becomes $1/\sqrt{81} = 1/9.\sqrt{81} = 1/9$, ■

Example 3.2.9 *(Application of the Mean Value Theorem) Show that*

$$|\sin b - \sin a| \leq |b - a|$$

for all $a, b \in \mathbb{R}$.

Solution Let $f(x) = \sin x$ on $[a,b]$. The function sin is continuous and differentiable on R. So the Mean Value Theorem finds $c : a < c < b$ such that

$$\sin b - \sin a = f(b) - f(a) = f'(c)(b-a) = \cos c\,(b-a).$$

Yet $|\cos c| \leq 1$, so the required result follows. ■

Method The following examples are all of showing $f(x) \geq g(x)$ for all $a \leq x \leq b$. Perhaps f is a 'complicated' function being bounded below by a 'simpler' g, or g is the complicated function being bounded above by a simpler f. In most cases define $F(t) = f(t) - g(t)$ and apply the Mean Value Theorem to F on $[a,x]$.

What is important in this solution is that the function F was **not** a function of x; I called the variable t. The x arises as the end point of the interval over which the Mean Value Theorem is applied.

Example 3.2.10

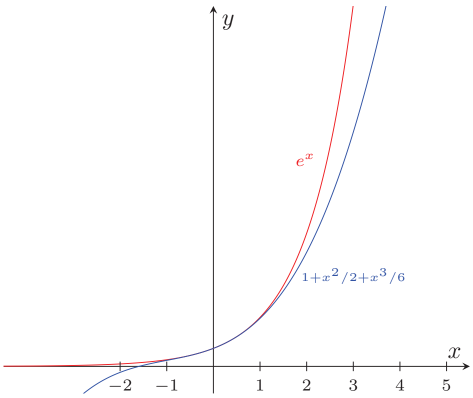
$$e^x > 1 + x + \frac{x^2}{2} \textit{ if } x > 0 \quad \textit{and} \quad e^x < 1 + x + \frac{x^2}{2} \textit{ if } x < 0.$$

Solution In tutorial.

Question Is it true that

$$e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

for all $x \in \mathbb{R}$? Left to interested student to prove (or see Appendix) though the following diagram suggests this is true:



Though I leave the above inequalities to you (and the tutorial) we will look at more complicated ones

Example 3.2.11

$$\frac{1}{1 - x + \dfrac{x^2}{3}} > e^x > \frac{1}{1 - x + \dfrac{x^2}{2}},$$

the first inequality for $0 < x < 1$*, the second for all* $x > 0$*.*

Solution of the right hand inequality. For $x > 0$ define

$$F(t) = \left(1 - t + \frac{t^2}{2}\right) e^t - 1$$

on $[0, x]$. This function is everywhere differentiable and thus continuous so we can apply the Mean Value Theorem to F on $[0, x]$. Thus there exists $c \in (0, x)$:

$$F(x) - F(0) = F'(c)\,(x - 0)\,.$$

Yet

$$F'(t) = (-1+t)\,e^t + \left(1 - t + \frac{t^2}{2}\right) e^t = \frac{t^2}{2} e^t > 0$$

for all t. Also $F(0) = 0$ so $F(x) > 0$ for all $x > 0$. Finally, to get stated result we need note that

$$1 - x + \frac{x^2}{2} = \left(1 - \frac{x}{2}\right)^2 + \frac{x^2}{4} \neq 0$$

for $x > 0$. This means we can divide by the polynomial.

The proof of the left hand inequality is left to the student. See Appendix■

> **It is important** to note that in all these examples the Mean Value Theorem was applied to $[0, x]$ and **not** $[0, 1]$.

In the next example, recall the Binomial Theorem which gives for $n \geq 1$ and $x > 0$:

$$(1+x)^n = 1 + nx + ... \geq 1 + nx,$$

the terms thrown away being positive since $x > 0$. This can be extended to some negative x as in

Example 3.2.12 *Bernoulli's Theorem (Application of the Mean Value Theorem) Let $x > -1$ and $n \geq 1$ be given, prove that*

$$(1+x)^n \geq 1 + nx.$$

Solution in Tutorial Define $F(t) = (1+t)^n - 1 - nt$ for $t \geq -1$. Given $x > -1$ apply the Mean Value Theorem to f on the closed interval

$$\begin{cases} [x,\, 0] & \text{if } x < 0 \\ [0,\, x] & \text{if } x \geq 0. \end{cases}$$

Then there exists a c between x and 0 such that

$$(1+x)^n - 1 - nx = F(x) - F(0) = F'(c)\,(x-0) = n\left((1+c)^{n-1} - 1\right) x. \quad (2)$$

We now have two cases.

The first case is when $x \geq 0$. If $x = 0$ our result follows with equality. If $x > 0$ then c satisfies $0 < c < x$ and so $(1+c)^{n-1} > 1$. Thus all terms on the right hand side of (2) are positive and thus $(1+x)^n - 1 - nx > 0$.

The second case is $x < 0$. This time c satisfies $x<c<0$ and so $(1+c)^{n-1} - 1 < 0$. Multiplying this by the **negative** x gives $\left((1+c)^{n-1} - 1\right) x > 0$ which in (2) again gives $(1+x)^n - 1 - nx > 0$.

Hence in both cases the required result follows. ■

Note an alternative proof can be given by induction on n.

No time to mention in lectures but, for an inequality involving the logarithm,

Example 3.2.13 *Prove that*

$$\ln(1+x) \geq xe^{-x}$$

for $x > 0$.

Solution Define $f(t) = \ln(1+t) - te^{-t}$, when $f(0) = \ln 1 - 0 = 0$. Given $x > 0$ apply the Mean Value Theorem to f on $[0, x]$ to find $c \in (0, x)$ such that

$$f(x) - f(0) = f'(c)(x - 0), \quad \text{i.e.} \quad f(x) = f'(c)\, x.$$

We look at this derivative for all $t > 0$ when

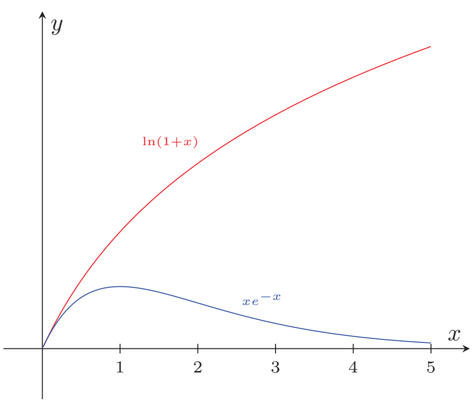
$$f'(t) = \frac{1}{1+t} - e^{-t} + te^{-t} = \frac{1 + e^{-t}(t^2 - 1)}{1+t} = e^{-t}\frac{e^t - 1 + t^2}{1+t}.$$

Here $e^t - 1 > 0$ and $t^2 > 0$ for $t > 0$. Hence $f'(t) > 0$ for all $t > 0$ and thus

$$f(x) = f'(c)\, x > 0$$

for $x > 0$. This gives the required result. ■

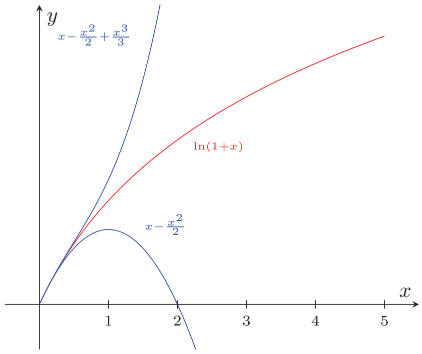
From the graph you can see this is only a good lower bound for x close to 0.



Note You will often see the bounds

$$x-\frac{x^2}{2}<\ln(1+x)<x-\frac{x^2}{2}+\frac{x^3}{3}.$$

The lower bound is weak when $x>2$, for it is then negative! The upper bound is a problem on the question sheet.



> **Advice for students.** These problems have reduced down to showing that $F(x)>F(a)$ for all x in an interval $[a,b]$. Do NOT apply the Mean Value Theorem on $[a,b]$, instead for each $x\in[a,b]$, apply the Mean Value Theorem on the *smaller* interval $[a,x]$. Many failed to do this in the exam.

Theorem 3.2.14 ***Increasing-Decreasing Theorem*** *Assume that f is differentiable on (a,b) and continuous on $[a,b]$.*

1) If $f'(x)>0$ for all $x\in(a,b)$ then f is strictly increasing on $[a,b]$.

2) If $f'(x)<0$ for all $x\in(a,b)$ then f is strictly decreasing on $[a,b]$.

3) If $f'(x)=0$ for all $x\in(a,b)$ then f is constant on $[a,b]$.

Proof *Not given in course* but the idea is, given *any* $a<x<y<b$, apply the Mean Value Theorem on $[x,y]$ to find $c\in(x,y)\subseteq(a,b)$ for which

$$f(y)-f(x)=f'(c)(y-x).$$

Since $y-x>0$ we can see that $f'(c)>0$ implies $f(y)>f(x)$ and $f'(c)<0$ implies $f(y)<f(x)$. ■

Note the converses of 1 and 2 are **not** true, i.e. it is possible for a function to be strictly increasing but $f'(x)$ is not positive for all x. Can you think of an example?

$$\boxed{\begin{array}{c}\forall x, f'(x)>0 \implies f \text{ is strictly increasing.}\\ f \text{ strictly increasing} \;\not\Longrightarrow\; \forall x, f'(x)>0.\end{array}}$$

Theorem 3.2.15 ***Cauchy's Mean Value Theorem*** *(1821) If f and g are differentiable on (a,b), continuous on $[a,b]$ and $g'(x) \neq 0$ for all x in (a,b) then there exists a $c : a < c < b$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Warning We **cannot** apply the earlier Mean Value Theorem to f and g separately since we might get different values for c for each function.

Note the left hand side is well-defined since $g'(c) \neq 0$ by assumption. But what of the right hand side? Could it be that $g(b) - g(a) = 0$? If $g(b) - g(a) = 0$ then by Rolle's Theorem applied to g we find c' (almost certainly different to the c above) for which $g'(c) = 0$. This contradicts our assumption, thus the right hand side is also well-defined.

Proof Define $h(x) = f(x) - \kappa g(x)$ with κ chosen such that $h(a) = h(b)$. That is,

$$\kappa = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Apply Rolle's Theorem to h on $[a,b]$ to find $c : a < c < b$ such that $h'(c) = 0$. That is

$$\frac{f'(c)}{g'(c)} = \kappa = \frac{f(b) - f(a)}{b - a}.$$

■

> **Advice for the exam.** When asked to state the Mean Value Theorem do NOT give Cauchy's Mean Value Theorem. If asked to state Cauchy's Mean Value Theorem do NOT give the Mean Value Theorem.

Example 3.2.16 *Assume that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$f(b) = f(a) + \frac{1}{2}\left(b^2 - a^2\right)\frac{f'(c)}{c}.$$

Solution for students (and tutorial). Ask yourself what needs to be chosen for $g(x)$. ■

Note this should be compared with what the Mean Value gives, namely there exists $c_1 \in (a,b)$ (possibly different to the c above) such that

$$f(b) = f(a) + (b - a) f'(c_1).$$

We now come to the proof of a result that students' love to use, even before differentiation has been defined!

Theorem 3.2.17 ***L'Hôpital's Rule*** *Suppose that f and g are both differentiable and $g'(x) \neq 0$ on some deleted neighbourhood of a. Then if $f(a) = g(a) = 0$ and $\lim_{x\to a} f'(x)/g'(x)$ exists, we have that $\lim_{x\to a} f(x)/g(x)$ exists and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof Let

$$\lambda = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

which we are told exists. We will verify the ε - δ definition that $\lim_{x\to a} f(x)/g(x) = \lambda$.

Let $\varepsilon > 0$ be given. Use this in the definition of $\lambda = \lim_{x\to a} f'(x)/g'(x)$ to find $\delta > 0$ such that if $0 < |x - a| < \delta$ then

$$\left| \frac{f'(x)}{g'(x)} - \lambda \right| < \varepsilon. \tag{3}$$

For such x consider

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{since } f(a) = g(a) = 0, \\ &= \frac{f'(c)}{g'(c)}, \end{aligned}$$

for some c between a and x, by Cauchy's Mean Value Theorem. (Note it may be that $x > a$ in which case Cauchy's Mean Value Theorem is applied on the interval $[a, x]$, whereas Cauchy's result is applied on $[x, a]$ if $x < a$.) Thus

$$\frac{f(x)}{g(x)} - \lambda = \frac{f'(c)}{g'(c)} - \lambda. \tag{4}$$

Here c lies *strictly* between a and x, in particular $c \neq a$ and c is closer to a than x. Hence $0 < |c - a| < |x - a| < \delta$. This means that (3) holds with x replaced by c, i.e.

$$\left| \frac{f'(c)}{g'(c)} - \lambda \right| < \varepsilon.$$

Then (4) gives

$$\left| \frac{f(x)}{g(x)} - \lambda \right| = \left| \frac{f'(c)}{g'(c)} - \lambda \right| < \varepsilon.$$

Summing up we have shown that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then

$$\left| \frac{f(x)}{g(x)} - \lambda \right| < \varepsilon.$$

Thus we have verified the definition that $\lim_{x \to a} f(x)/g(x) = \lambda$. ■

> **Advice for the exam.** If asked to *evaluate* a limit by verifying the definition, do **not** use the limit laws. If asked to *evaluate* a limit without restriction on the method used and you use a limit law *tell me* the rule being a used.

Example 3.2.18 *Prove*

$$\lim_{\theta \to 0} \frac{\sin \theta - \theta}{\theta^3} = -\frac{1}{6}$$

Solution Left to student ■

There are many variants of L'Hôpital's Rule.

i) The theorem remains true for $a = +\infty$ or $-\infty$. If we restrict to $x \in \mathbb{N}$, the case $a = +\infty$ will give us L'Hôpital's Rule for sequences as seen in MATH10242.

ii) The theorem remains true for limits $x \to a^+$ or $x \to a^-$.

iii) The theorem remains true if $\lim f(x) = \lim g(x) = \pm\infty$, along with the variants in (i) and (ii).